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Some Independence Results Related to the Kurepa Tree

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Abstract By an ω_1-tree we mean a tree of power ω_1 and height ω_1. Under the assumption of *CH* plus $2^{\omega_1} > \omega_2$ we call an ω_1-tree a Jech–Kunen tree if it has κ many branches for some κ strictly between ω_1 and 2^{ω_1}. We call an ω_1-tree being ω_1-anticomplete if it has more than ω_1 many branches and has no subtrees which are isomorphic to the standard ω_1-complete binary tree. In this paper we prove that: (1) It is consistent with *CH* plus $2^{\omega_1} > \omega_2$ that there exists an ω_1-anticomplete tree but no Jech–Kunen trees or Kurepa trees; (2) It is independent of *CH* plus $2^{\omega_1} > \omega_2$ that there exists a Jech–Kunen tree without Kurepa subtrees; (3) It is independent of *CH* plus $2^{\omega_1} > \omega_2$ that there exists a Kurepa tree without Jech–Kunen subtrees. We assume the existence of an inaccessible cardinal in some of our proofs.

Let T be a tree. For an ordinal α, T_α is the α-th level of T and $T|\alpha = \bigcup_{\beta<\alpha} T_\beta$. Let $ht(T)$, the height of T, be the smallest ordinal λ such that $T_\lambda = \emptyset$. By a branch of T we mean a linearly ordered subset of T which intersects every nonempty level of T. Let $\mathcal{B}(T) = \{B : B \text{ is a branch of } T\}$. For a $t \in T$ let $T(t) = \{s \in T : s \text{ and } t \text{ are comparable}\}$.

Let T be a tree. We recall that:

T is an ω_1-tree if $|T| = \omega_1$ and $ht(T) = \omega_1$. Without loss of generality we sometimes assume that $\langle T, \leq_T\rangle = \langle \omega_1, \leq_T\rangle$ with unique root 0 if T is an ω_1-tree.

An ω_1-tree T is called a *Kurepa tree* if $|T_\alpha| < \omega_1$ for any $\alpha < \omega_1$ and $|\mathcal{B}(T)| > \omega_1$.

An ω_1-tree T is called a *Jech–Kunen tree* if $\omega_1 < |\mathcal{B}(T)| < 2^{\omega_1}$.

T' is a *subtree of* T if $T' \subseteq T$ and $\leq_{T'} = \leq_T \cap T' \times T'$ (T' inherits the order of T). For an ordinal λ we call $\langle 2^{<\lambda}, \subseteq\rangle$ a *standard* λ-*complete binary tree.* A tree is called a λ-*complete binary tree* if it is isomorphic to $\langle 2^{<\lambda}, \subseteq\rangle$. A subtree T' of T is called *closed downward* if for any $t' \in T'$, $\{t \in T : t <_T t'\} \subseteq T'$.

An ω_1-tree T is called an ω_1-*anticomplete tree* if $|\mathcal{B}(T)| > \omega_1$ and T has no ω_1-complete binary subtrees.

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