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On Cofinal Extensions of Models of Fragments of Arithmetic

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Abstract We present a model-theoretic proof of Motohashi's preservation theorem for cofinal extensions, and examine various criteria for a model of a fragment of PA to have a proper elementary cofinal extension. Using these criteria we answer a question of Roman Kossak's, exhibiting for each $n \geq 0$ countable models M and N of $I\Sigma_n + \exp + \neg B\Sigma_{n+1}$ such that: (i) M has no proper elementary cofinal extensions and (ii) N does have proper elementary cofinal extensions.

Introduction Let $\mathcal{L}_A$ be the usual first-order language of arithmetic with nonlogical symbols $0, 1, +, \cdot, <$, and let PA^- be the $\mathcal{L}_A$-theory of the nonnegative parts of discretely ordered rings. The theories $I\Delta_0 + \exp$, $I\Sigma_n$, and $B\Sigma_n$ ($n \in \mathbb{N}$) are the usual fragments of Peano Arithmetic (PA). More specifically, $I\Sigma_n$ is axiomatized by PA^- together with the scheme of *Σ_n-induction*,

$$\forall \bar{a}(\theta(0,\bar{a}) \wedge \forall x(\theta(x,\bar{a}) \rightarrow \theta(x+1,\bar{a})) \rightarrow \forall x \theta(x,\bar{a})),$$

for all Σ_n formulas $\theta(x,\bar{a})$ (see Paris & Kirby [8]). The theory $I\Delta_0 + \exp$ is $I\Delta_0$ ($=I\Sigma_0$) together with a single axiom exp stating that the exponential function x^y is total (see Gaifman & Dimitracopoulos [2] for details in how this can be expressed in $\mathcal{L}_A$). The theory $B\Sigma_n$ is $I\Delta_0$ together with the scheme of *Σ_n-collection*,

$$\forall \bar{a}, t(\forall x < t \exists y \theta(x,y,\bar{a}) \rightarrow \exists z \forall x < t \exists y < z \theta(x,y,\bar{a})),$$

for all Σ_n formulas $\theta(x,y,\bar{a})$ (see [8]). With a certain convenient abuse of notation, we will write '$M \models I\Sigma_n + \neg B\Sigma_{n+1}$' to mean '$M \models I\Sigma_n$ and $M \not\models B\Sigma_{n+1}$', similarly for $B\Sigma_n + \neg I\Sigma_n$. Parsons [9] showed that $I\Sigma_{n+1} \vdash B\Sigma_{n+1} \vdash I\Sigma_n$ for all $n \geq 0$, and that models of $I\Sigma_n + \neg B\Sigma_{n+1}$ exist for all $n \geq 0$; Paris and Kirby [8] and (independently) Lessan [5] showed that models of $B\Sigma_n + \neg I\Sigma_n$ exist for all $n \geq 1$.

If M and N are models of PA^- and $M \subseteq N$ we say M is *cofinal in N*, $M \subseteq_{\mathrm{cf}} N$, iff $\forall a \in N \, \exists b \in M \, (N \models b > a)$; N is an *end-extension* of M, M is an initial

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