Notre Dame Journal of Formal Logic
Volume 32, Number 2, Spring 1991

The Cardinality of Powersets in Finite Models of the Powerset Axiom

ALEXANDER ABIAN and WAEL A. AMIN

Abstract It is shown that in a finite model of the set-theoretical Powerset axiom a set s and its powerset $\mathcal{P}(s)$ have the same number of elements. Additional results are also derived.

Let (F,ϵ) be a finite model of the set-theoretical Powerset axiom, i.e., in (F,ϵ) every set has a powerset.

For instance, let us consider the finite model (M,ϵ) whose domain consists of the four sets a,b,c,d and where the ϵ-relation is defined by:

(1) $a = \{b\}, \qquad b = \{a\}, \qquad c = \{a,b,c\}, \qquad d = \{a,b,c,d\}.$

It can be readily verified that (M,ϵ) is a model of the Powerset axiom. To this end, we have only to verify that every one of the sets a,b,c,d of the model (M,ϵ) has a powerset in (M,ϵ). For instance, to show that the powerset $\mathcal{P}(c)$ of c exists in (M,ϵ), we must show that all the subsets of c which exist in (M,ϵ) are collected by a set of (M,ϵ). As (1) shows, $c = \{a,b,c\}$ and therefore, from the point of view of the standard ZF set theory, c has $2^3 = 8$ subsets given by: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$. On the other hand, as (1) shows, of these 8 subsets of c only 3, namely $\{a\}$, $\{b\}$, $\{a,b,c\}$ are present in the model (M,ϵ). Again, as (1) shows, these 3 sets are respectively b,a,c and are collected in the model (M,ϵ) by the set c. Thus, we conclude that c is the powerset of c in the model (M,ϵ).

We observe that in the standard ZF set theory if a set has n elements then it has 2^n subsets. This is due to the fact that besides the Powerset axiom, ZF has other axioms which imply the existence of 2^n subsets for a set with n elements. By contrast, here we are considering finite set theoretical models and only the Powerset axiom, and we prove that in such models a set with n elements has n subsets.

Received September 30, 1989; revised January 3, 1990