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Notre Dame Journal of Formal Logic
Volume 32, Number 2, Spring 1991

Embedding Brouwer Algebras in the Medvedev Lattice

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Abstract We prove various results on embedding Brouwer algebras in the Medvedev lattice. In particular, we characterize the finite Brouwer algebras that are embeddable in the Medvedev lattice.

1 Introduction The following definition is fundamental throughout the paper:

Definition 1.1 Let $\mathfrak{L} = \langle L, \vee, \wedge, 0, 1\rangle$ be a distributive lattice with 0,1 and let $\leq$ be the partial ordering relation of $\mathfrak{L}$. Then $\mathfrak{L}$ is a *Brouwer algebra* if $\mathfrak{L}$ can be given a binary operation $\rightarrow$ such that, for every $a, b, c \in L$,

$$b \leq a \vee c \Leftrightarrow a \rightarrow b \leq c.$$

(Notice that this is equivalent to saying that the set $\{c \in L : b \leq a \vee c\}$ has a least element and this least element equals $a \rightarrow b$.)

Also, we say that a distributive lattice with 0,1 is a *Heyting algebra* if the dual of $\mathfrak{L}$ is a Brouwer algebra (for details on Heyting algebras see e.g. Balbes and Dwinger [1]). Heyting algebras are often called pseudo-Boolean algebras (see e.g. Rasiowa [10]). In the remainder of the paper, we will often use without further comment the fact that every finite distributive lattice with 0,1 is a Brouwer algebra (also, a Heyting algebra).

Now, let $\mathfrak{M}$ be the Medvedev lattice (see Medvedev [7] and Rogers [11]). In Sorbi [13] we show that $\mathfrak{M}$ is not a Heyting algebra. On the other hand, it is known ([7]; see also [11], Theorem 13.XXIV, for a proof) that $\mathfrak{M}$ is a Brouwer algebra. In this paper we show that as a Brouwer algebra $\mathfrak{M}$ is in fact a fairly rich one, by proving various embedding results. In particular, we obtain a characterization of the finite Brouwer algebras that are embeddable in $\mathfrak{M}$, thus extending a similar embedding result proved in Skvortsova [12]. Among the consequences of this result is also a proof (see Corollary 2.8 below) of the fact that the set of identities of $\mathfrak{M}$ (in the sense of [11], §13.7, i.e. the propositional

Received September 25, 1989; revised January 11, 1990