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# Expressive Completeness and Decidability

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**Abstract** Under what conditions is the expressive completeness of a set of connectives decidable? The answer is shown to depend crucially upon how the set is encoded as input to a Turing machine.

After Żliński showed in 1924 that | and ↓ are the only dyadic connectives adequate for expressive completeness, one would have thought that there was nothing left to say on the topic. Well, almost. Let **C** be a set of (two-valued) truth-functional connectives. Under what conditions on **C** is it decidable whether **C** is expressively complete? And under what conditions is it decidable whether a given connective is definable in terms of the members of **C**?

If only because of cardinality considerations, there is no decision procedure for expressive completeness for all **C**. In the special case where **C** is finite, however, there is such a procedure. This follows from an old, and relatively obscure, result of Post's. Given the idiosyncratic nature of the work in which the result is buried, it might be useful to restate it here.

Two rows $i$ and $j$ of a truth table are *mirror images* if every variable that is T on $i$ is F on $j$, and vice versa. The table is then *self-dual* if any two lines that are mirror images have different outputs.

A variable $p$ is *redundant* in a given table if any two rows that differ only in the value assigned to $p$ have the same output. Likewise, $p$ is *decisive* if any two rows that differ only in the value assigned to $p$ have different outputs. The table is then *alternating* if every variable is either redundant or decisive.

Finally, a truth table is *pairwise local* if, for any row $i$ with output T and row $j$ with output F, there is a variable that is T on $i$ and F on $j$.

**Theorem 1** (Post [2]) **C** *is expressively complete if and only if, for each of the following conditions, there is a connective in* **C** *whose truth table satisfies that condition*:

*Thanks are due an anonymous referee for bringing the work of Kudielka and Oliva [1] and Post [2] to our attention. The strategy of our second proof of decidability is essentially sketched in [1].

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