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Post's Functional Completeness Theorem

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Abstract The paper provides a new proof, in a style accessible to modern logicians and teachers of elementary logic, of Post's Functional Completeness Theorem. Post's Theorem states the necessary and sufficient conditions for an arbitrary set of (2-valued) truth functional connectives to be expressively complete, that is, to be able to express every (2-valued) truth function or truth table. The theorem is stated in terms of five properties that an arbitrary connective may have, and claims that a set of connectives is expressively complete iff for each of the five properties there is a connective that lacks that property.

Everyone knows the technique whereby, given an arbitrary (2-valued) truth table, one can construct a conjunctive (or disjunctive) normal form formula (using only connectives from $\{\vee,\wedge,\sim\}$) which has exactly that truth table. This proves that the set of connectives $\{\vee,\wedge,\sim\}$ is *functionally complete*: any (2-valued) truth table can be constructed from them. Everyone also knows the definitions of $\wedge$ in terms of $\{\vee,\sim\}$ and of $\vee$ in terms of $\{\wedge,\sim\}$. This shows that $\{\wedge,\sim\}$ and $\{\vee,\sim\}$ are also functionally complete sets of connectives. Everyone also knows that the sheffer stroke functions, $\uparrow$ and $\downarrow$, are each functionally complete. Most everyone knows that $\{\rightarrow,\mathbf{F}\}$ is functionally complete and that $\{\rightarrow,\underline{\vee}\}$ is functionally complete (**F** is the constant-false truth function, $\underline{\vee}$ is "exclusive or"). Some people, having worked through Church ([1], p. 131f.), even know that $\{[\ ],\mathbf{T},\mathbf{F}\}$ is functionally complete ([ ] is the ternary connective of "conditional disjunction": $[p,q,r]$ means "if q, then p else r"). However, what is not generally known is why these things are so. What is it about these particular sets of connectives that makes them functionally complete while (say) $\{\leftrightarrow,\sim\}$ is not functionally complete?

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