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The Dual Cantor-Bernstein Theorem and the Partition Principle

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Abstract This paper examines two propositions, the Dual Cantor-Bernstein Theorem and the Partition Principle, with respect to their logical interrelationship and their history. It is shown that the Refined Dual Cantor-Bernstein Theorem is equivalent to the Axiom of Choice.

1 Introduction We first recall some standard notation. If $x \leq y$ means that there is an injection $f: x \to y$, the dual relation $x \leq^* y$ is taken to mean that, if x is nonempty, there is a surjection $g: y \to x$. Analogously, $x < y$ means that $x \leq y$ and not $y \leq x$, while $x <^* y$ means that $x \leq^* y$ and not $y \leq^* x$. Letting $x \approx y$ mean that there is a bijection $f: x \to y$, we can express the Cantor-Bernstein Theorem as the proposition that if $x \leq y$ and $y \leq x$, then $x \approx y$. Likewise, the Dual Cantor-Bernstein Theorem (CB*) states that if $x \leq^* y$ and $y \leq^* x$, then $x \approx y$. The Partition Principle (PP) connects $\leq$ and $\leq^*$ by stating that $x \leq^* y$ implies $x \leq y$.

Neither the Dual Cantor-Bernstein Theorem nor the Partition Principle can be proved in Zermelo-Fraenkel set theory (ZF), but both are theorems if the Axiom of Choice (AC) is permitted. It is easily seen that CB* follows from PP in ZF, by means of the Cantor-Bernstein Theorem, and also that the converse of PP is a theorem of ZF. Now the Trichotomy of Cardinals (TC) states that, for all x and y, $x < y$ or $y < x$ or $x \approx y$. Likewise, the Dual Trichotomy of Cardinals (TC*) states that $x <^* y$ or $y <^* x$ or $x \approx y$. It turns out that TC is equivalent to AC (Hartogs [5]), and so is TC* (Lindenbaum and Tarski [9]; Sierpiński [18]).

It will be useful to introduce a certain refinement for each of CB*, PP, AC, and TC. The $\aleph_\alpha$-Dual Cantor-Bernstein Theorem ($\aleph_\alpha$-CB*) states that, for every x, if $x \leq^* \aleph_\alpha$ and $\aleph_\alpha \leq^* x$, then $x \approx \aleph_\alpha$. Analogously, the $\aleph_\alpha$-Partition Principle ($\aleph_\alpha$-PP) states that, for every x, if $\aleph_\alpha \leq^* x$, then $\aleph_\alpha \leq x$. It is clear that, for each α, $\aleph_\alpha$-PP implies $\aleph_\alpha$-CB*. In a similar vein, $\aleph_\alpha$-AC states that if $x \approx \aleph_\alpha$, then there is a function f such that $f(y) \in y$ for every nonempty member y of

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