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Two Hypergraph Theorems Equivalent to BPI

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Abstract Techniques originally developed for establishing NP-Completeness are adapted to prove that two compactness theorems concerning hypergraphs are equivalent to the Prime Ideal Theorem for Boolean algebras (BPI). In addition, some possible connections between NP-Completeness and BPI are explored.

1 Introduction We introduce two combinatorial compactness principles and show them to be in the large class of statements known to be equivalent in ZF set theory to BPI, the Prime Ideal Theorem for Boolean algebras (see, for example, [1], [2], [3], [4], [10]–[20], and [22] for other statements in this class). Both are about hypergraphs and were suggested by two NP-Complete decision problems considered by Schaefer [21]. In fact there seems to be an intimate connection between BPI and NP-Completeness; a major aim of this paper is to explore this connection.

2 A logical compactness theorem One of the more useful versions of BPI is the Compactness Theorem for propositional logic, which states that a set of propositional formulas is satisfiable if every finite subset is satisfiable. The equivalence of the Compactness Theorem for propositional logic and BPI was first proved by Henkin [10]. Here we shall need a restricted version—when all the formulas are disjunctions of at most three literals (a literal is a statement letter or its negation). This restricted version of the Compactness Theorem will be referred to as 3-SAT and our first task is to show that 3-SAT is equivalent to BPI.

Theorem 1 3-SAT $\Leftrightarrow$ BPI.

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