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The Modal Logic of Pure Provability

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Abstract We introduce a propositional modal logic PP of "pure" provability in arbitrary theories (propositional or first-order) where the $\Box$ operator means "*provable in all extensions*". This modal logic has been considered in another guise by Kripke. An axiomatization and a decision procedure are given and the $\Box\Diamond$ subtheory is characterized.

1 Introduction This paper discusses a modal logic PP of *pure* provability; that is to say, of provability in arbitrary theories (propositional or first-order). The modal formula $\Box\phi$ is intended to mean "ϕ is provable in all possible extensions of the present theory"; the subtleties arise in the interpretation of iterated modalities. The modal theory we are studying is very different from the provability interpretation of Solovay [6]; we allow theories which are much weaker than Peano Arithmetic and which may be unable to formalize metamathematics. Our theory PP was briefly mentioned by Kripke [3]; however, Kripke's aims were somewhat different from ours and he did not explore PP in depth. The aim and motivation of this paper is to give a modal theory of provability; Kripke's program (fulfilled by Solovay) was to give a provability interpretation of modal logic.

It turns out that the modal theory PP of pure provability is somewhat pathological; most notably, PP is not closed under the rule of substitution of arbitrary wff's for propositional variables. On the other hand, PP does satisfy all consequences of S4 and McKinsey's axiom (of S4.1). Furthermore, we provide

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