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The Nonaxiomatizability of $L(Q^2_{\aleph_1})$ by Finitely Many Schemata

SAHARON SHELAH* and CHARLES STEINHORN**

Abstract Under set-theoretic hypotheses, it is proved by Magidor and Malitz that logic with the Magidor–Malitz quantifier in the $\aleph_1$-interpretation is recursively axiomatizable. It is shown here, under no additional set-theoretic hypotheses, that this logic cannot be axiomatized by finitely many schemata.

Magidor and Malitz [2] introduced the n-variable-binding quantifiers Q^n. The language $L(Q^n)$ is formed by adding Q^n to first-order predicate logic. For an infinite cardinal κ, $Q^n x_1 x_2 \ldots x_n \varphi$ may be assigned the so-called κ-interpretation in a structure $\mathfrak{M}$, wherein $Q^n x_1 \ldots x_n \varphi$ is satisfied if there exists an $A \subseteq \mathfrak{M}$ of power κ that is homogeneous for φ, i.e., for any $a_1, \ldots, a_n \in A$, $\varphi(a_1, \ldots, a_n)$ holds in $\mathfrak{M}$. Among many other results Magidor and Malitz establish, under the set-theoretic axiom $\diamondsuit_{\aleph_1}$, a completeness theorem for $L(Q^n)$ in the $\aleph_1$-interpretation (hereafter $L(Q^n_{\aleph_1})$). Unfortunately, the complete axiom system for $L(Q^n_{\aleph_1})$ exhibited in [2] lacks the simplicity of, e.g., Keisler's set of axioms for $L(Q^1_{\aleph_1})$ (cf. [1]).

This paper, a sequel to [3], demonstrates that this failure of simplicity is not without reason. It will be shown here, without additional set-theoretic hypotheses, that $L(Q^2_{\aleph_1})$ cannot be axiomatized by finitely many schemata. Even more strongly, we prove:

Theorem 1 *No collection of axiom schemata of bounded quantifier depth suffices to axiomatize* $L(Q^2_{\aleph_1})$.

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