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The Completeness of Provable Realizability

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Abstract Let A be a propositional formula and $r_A[x]$ express in the predicate logic the statement "x realizes A". We prove that the classical derivability of $r_A[t]$ for a lambda term t implies the intuitionistic derivability of A for the formulas A in the languges $(\supset, \wedge, \neg)$ and $(\supset, \&)$, where & is the so-called strong conjunction.

Intuitionistic logical connectives are often supposed to be determined by semantical constructions. For every such (binary) connective C it should be determined how the construction proving (or justifying) $C(A,B)$ is composed from the ones justifying A, B. This approach is traceable to Brouwer and was clearly formulated in [2] and [6]. Various notions of realizability beginning with [4] can be thought of as formalizations of these ideas. We follow [7] in formalizing further and show that the provability of the formula A in the intuitionistic propositional calculus with implication, negation, conjunction $\wedge$, and strong conjunction & (see below) coincides with the classical provability of the formula $r_A[t]$, expressing in the language of the predicate calculus the statement: "the λ-term t realizes the formula A". The main results of this paper were announced in [8].

Adding the disjunction connective changes the situation drastically: the familiar formula constructed by G. Rose is unprovable but realizable by a suitable if-then-else term.

Strong conjunction & is determined by the stipulation: x realizes $A \,\&\, B$ if x realizes both A and B.

The definition of the formula $r_A[t]$ and the statement of the problem are taken essentially from [7]. Nevertheless, the propositional calculus formulated in [7] for implication and strong conjunction is incomplete, contrary to the conjecture made in [7]: consider the formula $((A \supset B) \,\&\, D) \supset (((A \,\&\, C) \supset B) \,\&\, D)$. It is realizable by the term $\lambda x.x$ but unprovable in the calculus (see [7]).

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