"Pathologies" in Two Syntactic Categories of Partial Maps

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Introduction In [2], Di Paola and Heller introduce the dominical categories and the recursion categories in order to find an algebraic (or, better, category-theoretic) approach to recursion theory. The authors show that the basic part of recursion theory can be based on a few category-theoretic axioms and prove, by means of several relevant examples, that their approach is suitable not only for classical recursion theory, but also for a broad class of situations. The authors suggest that their approach could provide an algebraic version of Gödel-Rosser's incompleteness theorems, and provide a first, but relevant, step in this direction, showing that in any recursion category satisfying some natural conditions there are creative and effectively inseparable domains.

To carry their project one step further, it seems quite natural to study categories of partial maps from a syntactic point of view. This study has also been suggested by Di Paola and Heller. The most interesting category of this kind seems to me to be the one whose objects and morphisms are defined as follows:

1. The class \( \text{Ob} \) of objects is the smallest class \( C \) such that \( \omega \in C, A, B \in C \) implies \( A \times B \in C \) and \( A + B \in C \) (where \( A + B = A \times \{0\} \cup B \times \{1\} \) is the disjoint union of \( A \) and \( B \)).

2. The morphisms from \( A \) to \( B \) are the Gödel numbers of partial recursive functions from \( A \) to \( B \) modulo provable equality in PA, i.e., the equivalence classes of natural numbers with respect to the equivalence \( \equiv \) defined by \( n \equiv m \text{ iff } PA \vdash \forall x[(\varphi_n x \uparrow \leftrightarrow \varphi_m x \uparrow) \land (\varphi_n x \downarrow \rightarrow \varphi_m x = \varphi_n x)] \).

Here we are using notation from [8]; moreover, if \( A \in \text{Ob} \), the elements of \( A \) can be naturally coded by natural numbers, and we can identify each \( a \in A \) with its code in \( \omega \). We denote this category by \( S \).

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