

Notes on Modal Definability

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1 Introduction This paper contains a few observations on the definability of frame classes in modal logic, utilizing current algebraic methods in the area. For technical background, see [3], [4].

Possible worlds frames induce modal algebras of their subsets, and conversely, modal algebras can be represented as frame-induced set algebras by the Stone ultrafilter representation. This back-and-forth connection allows for a transfer of existing definability results in Universal Algebra to the model theory of possible worlds frames. One notable result is that, in translating the Birkhoff characterization of equational varieties, if a frame F validates the full modal theory of some frame class K , then the following structural connection exists: The ‘ultrafilter extension’ $ue(F)$ is a generated subframe of a p -morphic image of an ultrafilter extension of some disjoint union of frames in K . Several well-known theorems on the modal definability of frame classes have been deduced from this and similar observations. Here we shall take a closer look at the structure of the *ultrafilter extensions* involved (Section 2), deriving some additional *definability results* (Section 3). Then we particularize the theory to an important special case, viz. that of *finite frames*, which turns out to require additional techniques (Section 4). Finally, another specialization is considered, to the case of singleton classes K , i.e. to the study of *modal equivalence* between frames (Section 5).

The notion of an ultrafilter extension and its various uses forms a red thread through this report—which is otherwise a loose collection of results ‘rounding out’ the existing literature.

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