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On Propositional Quantifiers in Provability Logic

SERGEI N. ARTEMOV and LEV D. BEKLEMISHEV

Abstract The first order theory of the Diagonalizable Algebra of Peano Arithmetic (DA(PA)) represents a natural fragment of provability logic with propositional quantifiers. We prove that the first order theory of the 0-generated subalgebra of DA(PA) is decidable but not elementary recursive; the same theory, enriched by a single free variable ranging over DA(PA), is already undecidable. This gives a negative answer to the question of the decidability of provability logics for recursive progressions of theories with quantifiers ranging over their ordinal notations. We also show that the first order theory of the free diagonalizable algebra on n independent generators is undecidable iff $n \neq 0$.

1 Introduction Gödel was probably the first to consider the *provability interpretation of modal logic*: according to it the modality $\Box$ is understood as the standard arithmetical Σ_1-predicate $Pr(\cdot)$ expressing provability in Peano arithmetic PA (cf. [15]). A complete axiomatization together with a decision procedure for the propositional modal logic of provability was given in Solovay [21]. On the other hand, it was shown in Artemov [2] and Vardanyan [23] that predicate provability logic has no r.e. axiom systems.

One of the most interesting remaining problems in this area is that of axiomatizability and decidability of the Provability Logic with Propositional Quantifiers (PLPQ). Informally speaking, PLPQ is the set of all formulas in a modal language with quantifiers over propositions, which are true in the standard model of PA under the interpretation of propositional variables as (the Gödel numbers of) arbitrary arithmetic sentences, and $\Box$ as $Pr(\cdot)$. For example, PLPQ contains the "usual" Hilbert–Bernays derivability conditions

$$\forall p, q\ \Box(\Box(p \to q) \to (\Box p \to \Box q)), \quad \forall p\ \Box(\Box p \to \Box\Box p),$$

formalized Löb's theorem

$$\Box\, \forall p\ (\Box(\Box p \to p) \to \Box p)$$

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