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The Strength of the Δ-system Lemma

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Abstract The delta system lemma is not provable in set theory without the axiom of choice nor does it imply the axiom of choice.

1 Introduction A *Δ-system* $\mathcal{G}$ is a collection of sets such that there is a set r with the property that $(\forall A \in \mathcal{G})(\forall B \in \mathcal{G})(A \neq B \Rightarrow A \cap B = r)$. r is called the *root* of $\mathcal{G}$. The *Δ-system lemma* is the statement:

ΔSL *For every uncountable collection $\mathcal{F}$ of finite sets there is an uncountable subcollection $\mathcal{G}$ of $\mathcal{F}$ which forms a Δ-system.*

ΔSL is provable in Zermelo Fraenkel set theory (ZF) with the axiom of choice (AC) as shown by Kunen [3], [4]. We will investigate the strength of ΔSL in ZF (without the axiom of choice). In this theory there are two possible definitions of *X is uncountable*: $|X| \not\leq \aleph_0$ or $\aleph_0 < |X|$. These definitions are equivalent if AC is assumed. In Section 2 below we will use the first definition exclusively. In Section 3 we will investigate the consequences of using the second definition.

We will also refine ΔSL in the following way: $\Delta SL(n)$ will denote, for each positive integer n, the Δ-system lemma for families of n-element sets. We note that $\Delta SL(1)$ is trivially true. Our main goal will be to prove that for any integer $n \geq 2$, $\Delta SL(n)$ is equivalent to ΔSL and also to the conjunction of the two statements:

CU *The union of a countable collection of countable sets is countable.*

and

PC *Every uncountable collection of countable sets has an uncountable subcollection with a choice function.*

2 Using the first definition of uncountable We begin with:

Lemma 2.1 $ZF \vdash (\forall n \in \omega - \{0\})(\Delta SL(n+1) \Rightarrow \Delta SL(n))$.

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