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# Maximal Subgroups of Infinite Symmetric Groups

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**Abstract** We prove that it is consistent that there exists a subgroup of the symmetric group $\mathrm{Sym}(\lambda)$ which is not included in a maximal proper subgroup of $\mathrm{Sym}(\lambda)$. We also consider the question of which subgroups of $\mathrm{Sym}(\lambda)$ stabilize a nontrivial ideal on $\lambda$.

***1 Introduction*** The work in this paper was motivated by the following question, which was raised by Peter Neumann. If $\lambda \geq \omega$, does every proper subgroup of $\mathrm{Sym}(\lambda)$ lie in a maximal subgroup of $\mathrm{Sym}(\lambda)$? While a positive answer seems very unlikely, all of the results up to this point have concerned sufficient conditions for a subgroup $G < \mathrm{Sym}(\lambda)$ to lie in a maximal subgroup of $\mathrm{Sym}(\lambda)$. For example, the main theorem in MacPherson and Praeger [3] states that if $G < \mathrm{Sym}(\omega)$ is not highly transitive, then $G$ is contained in a maximal subgroup. In Section 2, we shall prove the following result.

**Theorem 1** ($F_\lambda$) *There exists a subgroup* $G < \mathrm{Sym}(\lambda)$ *such that the set* $\mathbb{L} = \{H \mid G \leq H < \mathrm{Sym}(\lambda)\}$ *is a well-ordering under inclusion of order-type* $2^\lambda$. *In particular, G is not contained in a maximal subgroup of* $\mathrm{Sym}(\lambda)$.

It is not known whether this theorem can be proved in ZFC. Our extra hypothesis $F_\lambda$ is the following statement. Let $\mathrm{Sym}_{<\lambda}(\lambda)$ be the group of all permutations $\pi$ of $\lambda$ such that $|\mathrm{Mov}(\pi)| < \lambda$, where $\mathrm{Mov}(\pi) = \{\alpha \mid \alpha^\pi \neq \alpha\}$. Let $S(\lambda) = \mathrm{Sym}(\lambda)/\mathrm{Sym}_{<\lambda}(\lambda)$.

($F_\lambda$) If $T < S(\lambda)$ is a subgroup with $|T| < 2^\lambda$, then there exists an element of infinite order $\pi \in S(\lambda) \setminus T$ such that $\langle T, \pi \rangle = T * \langle \pi \rangle$.

Here $*$ denotes the free product. We shall also show that $F_\lambda$ is consistent with but independent of ZFC.

Another result from [3] states that if $I$ is a nontrivial ideal on $\lambda$ which contains a set $X$ with $|X| = |\lambda \setminus X| = \lambda$, and $G \leq S_{\{I\}} = \{\pi \in \mathrm{Sym}(\lambda) \mid I^\pi = I\}$,

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