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On Potential Embedding and Versions of Martin's Axiom

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Abstract We give a characterization of versions of Martin's axiom and some other related axioms by means of potential embedding of structures.

1 Introduction Let A and B be structures. For a condition $\mathcal{E}$ on p.o.-sets (e.g., ccc, proper, $<\kappa$-closed, etc.) let us say that A is $\mathcal{E}$-potentially embeddable into B if there exists a p.o.-set P with the property $\mathcal{E}$ such that $\Vdash_P$ "A is embeddable into B". Similarly we shall say that A and B are $\mathcal{E}$-potentially isomorphic if there exists a p.o.-set P with the property $\mathcal{E}$ such that $\Vdash_P$ "$A \cong B$".

The notion of $(<\kappa,\infty)$-distributive-potentially isomorphism and $<\kappa$-closed-potentially isomorphism have been studied in Nadel and Stavi [7]. In Fuchino, Koppelberg, and Takahashi [4] a characterization of $(<\kappa,\infty)$-distributive-potentially isomorphism to a free Boolean algebra is given under certain set theoretic assumptions on κ.

In this note we shall consider the question if $\mathcal{E}$-potential embedding ($\mathcal{E}$-potential isomorphism) implies the real embedding (isomorphism).

The following examples suggest that this question is by no means trivial for some instances of A and B even when we consider the ccc as the condition $\mathcal{E}$. Example 1.1c is due to S. Kamo.

Example 1.1

(a) Let A be the subalgebra of the Boolean algebra $\wp(\omega_1)$ consisting of finite and co-finite subsets of ω_1. Assume that there exists a ccc Boolean algebra B which is not productively ccc. Let C be a ccc Boolean algebra such that $B \oplus C$ does not satisfy the ccc. By the ccc of B, A is not embeddable in B. But, since $\Vdash_{C^+}$ "B does not satisfy the ccc", we obtain the result that $\Vdash_{C^+}$ "A is embeddable into B".

This situation can also be coded in structures in a language with only a binary relation symbol: Let B and C be as above. Let D and E be the structures defined by

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