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Generic Models of the Theory of Normal Z-rings

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Abstract A normal **Z**-ring M is a discretely ordered ring, integrally closed in its fraction field and such that for each positive integer n, $M/nM \simeq \mathbf{Z}/n\mathbf{Z}$ as rings. Here we study some properties of finite generic normal **Z**-rings. We give a uniform universal definition of **N** in them. And we separate existentially closed normal **Z**-rings via generics.

1 Introduction and preliminaries Let $\mathcal{L}$ denote the first order language of ordered rings based on the symbols 0, 1, +, −, ·, <. The theory of normal **Z**-rings (NZR) consists of the following axioms:

(i) OR: the theory of ordered rings;
(ii) D: $\forall x \neg (0 < x < 1)$ (the discreteness of the order);
(iii) N: for each $n \in \mathbf{N}$

$$\forall z_1, \dots, z_n xy (x, y \neq 0 \wedge x^n + z_1 x^{n-1} y + \cdots + z_n y^n = 0 \rightarrow \exists w (x = wy))$$

and the **Z**-ring axioms:

(iv) Z: for each $n \in \mathbf{N}$, $n \neq 0$ $\forall x \forall yz (x = ny + z \wedge 0 \leq z < n)$.

The theory of normal **Z**-rings plays a relevant role in the study of the fragment of arithmetic Normal Open Induction (NOI). NOI is the $\forall\exists$-theory in the language $\mathcal{L}$ which consists of NZR together with

$$\forall \mathbf{x}((\theta(\mathbf{x},0) \wedge \forall y \geq 0(\theta(\mathbf{x},y) \rightarrow \theta(\mathbf{x},y+1)) \rightarrow \forall y \geq 0 \theta(\mathbf{x},y))$$

for every quantifier-free $\mathcal{L}$-formula $\theta(\mathbf{x},y)$ ($\mathbf{x}$ denotes an n-tuple $(x_1, \dots, x_n)$).

In [7] Shepherdson gave the following useful characterization of models of NOI:

> Let M be a normal discretely ordered ring. Then M is a model of NOI if and only if for every element α of the real closure of the fraction field of M there is an element a in M such that $|a - \alpha| < 1$.

From this several corollaries are deduced. Let us mention some of them. Let M be a model of NOI. Then every quantifier-free definable set in M is a finite

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