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LETTER TO THE EDITOR

Dear Editor,

On the covariances of outdegrees in random plane recursive trees

In 2005 Janson [3], extending the earlier work of Mahmoud *et al.* [4], established the joint asymptotic normality of the outdegrees of a random plane recursive tree (we refer to [3] for references, discussion, and statements, and to [2] for a much wider context). In particular, he gave the following formula for the entries of the limiting covariance matrix [3, Theorem 1.3]:

$$\tilde{\sigma}_{ij} = 2\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{(-1)^{k+l}}{k+l+4}\binom{i}{k}\binom{j}{l}\left(\frac{2(k+l+4)!}{(k+3)!(l+3)!} - 1 - \frac{(k+1)(l+1)}{(k+3)(l+3)}\right). \qquad (1)$$

Since this formula is not very convenient to work with (in particular the behavior of $\tilde{\sigma}_{ij}$ as i and/or j grow to ∞ is not immediately clear), we found it worthwhile to point out that it may be simplified considerably. Throughout, $(x)_m = x(x-1)\dots(x-(m-1))$ denotes the falling factorial.

Proposition 1. *For all integers $i \geq 0$, $j \geq 0$, we have*

$$\tilde{\sigma}_{ij} = \frac{16}{(i+3)_3(j+3)_3} - \frac{24}{(i+j+4)_4} \quad \text{if } i \neq j,$$
$$\tilde{\sigma}_{jj} = \frac{4}{(j+3)_3} + \frac{16}{(j+3)_3^2} - \frac{24}{(2j+4)_4}.$$

For the proof we will need two identities involving binomial coefficients that we present in the following two lemmas.

Lemma 1. *For all integers $k \geq 0$, $a \geq 0$, and $j \geq k$,*

$$\sum_{l=0}^{j}(-1)^l\binom{j}{l}\binom{k+l+a}{l+a} = \begin{cases} 0 & \text{if } j > k, \\ (-1)^j & \text{if } j = k. \end{cases}$$

Proof. This is a special case of equation (5.24) in [1] as we have found thanks to the encouragement by one of the referees to search for a source in the literature. It corresponds to $m = 0$ and $s = n + a$ in the notation used in [1]. However, to keep this letter self-contained we supply a short proof. We proceed by induction over k for all a and $j \geq k$. If $k = 0$ the equality holds for all $a \geq 0$ since its left-hand side is $(1-1)^j$ if $j > 0$ and 1 if $j = 0$. Assume that it holds for nonnegative integers up to k and all values of a and $j \geq k$. Let $a \geq 0$ be any integer. For $j \geq k+1$,

$$\sum_{l=0}^{j}(-1)^l\binom{j}{l}\binom{k+1+l+a}{l+a} = \frac{k+1+a}{k+1}\sum_{l=0}^{j}(-1)^l\binom{j}{l}\binom{k+l+a}{l+a}$$
$$+ \sum_{l=0}^{j}(-1)^l\left(\frac{j!}{l!(j-l)!}\right)\left(\frac{l(k+l+a)!}{(k+1)!\,(l+a)!}\right).$$

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