ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 6, 2007

ON A CONJECTURE OF KUFNER AND PERSSON

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ABSTRACT. The L^p-L^q boundedness of the (conjugate) Hardy operator $(L^*f)(x) = \int_x^\infty l(t,x)f(t)\,dt$ for the case $0 < q < 1$ has been studied and thereby a conjecture of Kufner and Persson is proved. An important role here is played by level type functions.

1. Introduction. The boundedness of the Hardy operator $(Hf)(x) = \int_0^x f(t)\,dt$ between weighted Lebesgue spaces $L^p((0,\infty),v)$ and $L^q((0,\infty),u)$, $p \in (1,\infty)$, $q \in (0,\infty)$, has been studied quite extensively during the last decades. Simultaneously, as a natural case, the corresponding conjugate Hardy operator $(H^*f)(x) = \int_x^\infty f(t)\,dt$ was also considered and studied. A good account of such work can be found in [**3, 4, 7**]. In order to obtain the boundedness of H^*, generally, two methods are employed. One is using the duality arguments and the other is by making suitable variable transformations in the Hardy inequality

$$\left(\int_0^\infty [(Hf)(x)]^q u(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p}.$$

In the case $1 < p$, $q < \infty$, the two methods yield the same necessary and sufficient conditions. However, when $0 < q < 1$, we cannot use duality arguments. Moreover, in this case, the proof requires quite a different approach than the other cases. This case is due to Sinnamon [**8**] who made use of "level functions" introduced by Halperin [**2**].

Further, Bloom and Kerman [**1**] and Oinarov [**5, 6**], see also [**3**], studied the boundedness of the generalized Hardy operator $(Lf)(x) = \int_0^x l(x,t)f(t)\,dt$ and its corresponding conjugate operator $(L^*f)(x) = \int_x^\infty l(t,x)f(t)\,dt$ involving the so-called "Oinarov kernel" $l(x,t)$. It

2000 AMS *Mathematics subject classification.* Primary 26D10, 46E35.
Keywords and phrases. Hardy operator, conjugate Hardy operator, kernel, level functions, boundedness.
Received by the editors on November 9, 2004, and in revised form on August 13, 2005.

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