ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 5, 2007

FIXED POINTS OF PLANAR HOMEOMORPHISMS OF THE FORM IDENTITY + CONTRACTION

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ABSTRACT. Let f be a planar homeomorphism which has the form Identity + Contraction. We prove the existence of a fixed point of f under some geometrical condition on an orbit of f. The paper improves the result of Aarao and Martelli and provides an example which shows that, in the given setting, the theorem cannot be made stronger.

1. Introduction. The purpose of this paper is to study the existence of fixed points of maps $F : U \to \mathbf{R}^2$, where U is an open subset of $\mathbf{R}^2$, $F(x) = x + K(x)$ and K is contraction, i.e., $\|K(x)-K(y)\| \leq k\|x-y\|$, $0 < k < 1$. Each such map is an orientation preserving homeomorphism, cf. [**1**]. On the other hand, it is known that for orientation preserving homeomorphisms the presence of a periodic orbit forces the existence of a fixed point, which is an equivalent of one version of Brouwer's lemma on translation arcs, cf. [**2, 3**–**6**]. In [**1**] a stronger result is proved for the class of maps under consideration. Let for the rest of the paper $x_n = F^{n-1}(x)$ and $B(x, r)$ be a closed ball centered at x with the radius r. The theorem of Aarao and Martelli proved in [**1**] is the following:

Theorem 1.1. *Assume that there is a finite sequence $\{x_1, \dots, x_{n+1}\}$ such that its convex hull C is contained in U. Then there exists a point y in C such that $K(y) = 0$ provided that there is a $w \in [x_n, x_{n+1}]$ such that*:

$$\|w - x_1\| \leq \sqrt{1-k^2}\|K(x_1)\|. \tag{1.1}$$

Aarao and Martelli suspected, cf. [**1**, page 21], that the inequality (1.1) is optimal, i.e., that there are maps without fixed points with

2000 AMS *Mathematics subject classification.* Primary 37B30, 37E30, 55M25.
Keywords and phrases. Fixed points, contraction, winding number, fixed point index, planar homeomorphisms.
Research of the first author supported by KBN grant No 2 P03A 04522.
Received by the editors on March 1, 2004, and in revised form on May 30, 2005.