CURVES IN BANACH SPACES–DIFFERENTIABILITY VIA HOMEOMORPHISMS

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ABSTRACT. We prove several results on curves $f : [0,1] \to X$, where X is an arbitrary real Banach space. They generalize theorems which were proved by Zahorski, Tolstov, Choquet and Bari in the case $X = \mathbf{R}^n$. First we give a complete characterization of those f that admit an equivalent parametrization which has a continuous derivative (respectively with continuous derivative which is non-zero everywhere or almost everywhere). Further we establish theorems characterizing curves allowing boundedly or finitely differentiable parametrizations (with almost everywhere nonzero derivative). As a tool, we prove versions of the aforementioned theorems for metric analogues of derivatives. Finally, we discuss the case of curves allowing almost everywhere differentiable parametrizations. We also answer several questions posed by Bruckner.

1. Introduction. We prove several results on curves $f : [0,1] \to X$, where X is an arbitrary real Banach space. Our results give a complete characterization of several situations when there exists an equivalent parametrization of a curve possessing various differentiability properties. They generalize theorems which were known (to our knowledge) for the case $X = \mathbf{R}^n$ only. For some proofs we need, besides the known methods used in the case $X = \mathbf{R}^n$ and results on metric differentiability of Lipschitz (and pointwise-Lipschitz) mappings (from [**10, 14**]), also some new ideas.

Our result on C^1-parametrizations (Theorem 3.1) generalizes a theorem of Tolstov [**17**] for curves with values in the Euclidean space $\mathbf{R}^n$. Note that in [**17**], only curves, which are non-constant on any interval, are considered, and that the result for real functions (possibly constant on an interval) was proved independently by Bruckner and

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AMS *Mathematics subject classification.* Primary 26A24, Secondary 26E20.
Keywords and phrases. Curves in Banach spaces, differentiable parametrizations, metrically differentiable functions, tangents, VBG_* functions.
The first author was supported in part by ISF. The second author was supported by the grants GAČR 201/03/0931 and MSM 0021620839.
Received by the editors on February 25, 2005, and in revised form on April 21, 2005.