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SOME MIXED-TYPE REVERSE-ORDER LAWS FOR THE MOORE-PENROSE INVERSE OF A TRIPLE MATRIX PRODUCT

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ABSTRACT. Using some rank formulas for partitioned matrices and outer inverses of a matrix, we derive necessary and sufficient conditions for a group of mixed-type reverse-order laws to hold for the Moore-Penrose inverse of a triple matrix product.

1. Introduction. Throughout this paper, A^*, $r(A)$ and $\mathcal{R}(A)$ denote the conjugate transpose, rank and range (column space) of a complex matrix A, respectively; $[\, A,\, B\,]$ denotes a row block matrix consisting of A and B.

Suppose A and B are two nonsingular matrices of the same size. Then the product AB is nonsingular, too, and the inverse of AB satisfies the ordinary reverse-order law $(AB)^{-1} = B^{-1}A^{-1}$. This law can be used to simplify various matrix expressions that involve inverses of matrix products. This formula, however, cannot trivially be extended to generalized inverses of matrix products. For an $m \times n$ matrix A, the Moore-Penrose inverse $A^{\dagger}$ of A is defined to be the unique solution of the following four Penrose equations

(i) $AXA = A$,

(ii) $XAX = X$,

(iii) $(AX)^* = AX$,

(iv) $(XA)^* = XA$.

For simplicity, let $E_A = I - AA^{\dagger}$ and $F_A = I - A^{\dagger}A$, which are two orthogonal projectors induced by A. A matrix X is called a generalized inverse of A, denoted by A^-, if it satisfies $AXA = A$, an outer inverse of A if it satisfies $XAX = X$, and a reflexive generalized inverse of A,

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