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SOME EXTENSIONS OF THE MARKOV INEQUALITY FOR POLYNOMIALS

D. DRYANOV, R. FOURNIER AND S. RUSCHEWEYH

ABSTRACT. Let $\mathbf{D}$ denote the unit disc of the complex plane and $\mathcal{P}_n$ the class of polynomials of degree at most n with complex coefficients. We prove that

$$\max_{z\in\partial\mathbf{D}}\left|\frac{p_k(z)-p_k(\bar z)}{z-\bar z}\right| \le n^{1+k}\max_{0\le j\le n}\left|\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}\right|,$$

where $p_0 := p$ belongs to $\mathcal{P}_n$ and for $k\ge 0$, $p_{k+1}(z) := zp_k'(z)$. We also obtain a new proof of a well-known inequality of Duffin and Schaeffer and sharpenings of some other classical inequalities.

Introduction. Let $\mathcal{P}_n$ be the class of polynomials

$$p(z)=\sum_{k=0}^{n} a_k(p)z^k$$

of degree at most n with complex coefficients. We define, together with $\mathbf{D} := \{z \mid |z|<1\}$,

$$\|p\|_{\mathbf{D}} := \max_{z\in\partial\mathbf{D}}|p(z)| \quad \text{and} \quad \|p\|_{[-1,1]} := \max_{-1\le x\le 1}|p(x)|.$$

The famous inequalities of, respectively, Bernstein and Markov state that for any $p\in\mathcal{P}_n$,

$$\|p'\|_{\mathbf{D}} \le n\|p\|_{\mathbf{D}} \tag{1}$$

and

$$\|p'\|_{[-1,1]} \le n^2\|p\|_{[-1,1]}, \tag{2}$$

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