ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 4, 2007

HOMOLOGY OF ZERO-DIVISORS

REZA AKHTAR AND LUCAS LEE

ABSTRACT. Let R be a commutative ring with unity. We define a semi-simplicial abelian group based on the structure of the semigroup of ideals of R and investigate various properties of the homology groups of the associated chain complex.

1. Introduction. Let R be a commutative ring with unity. The set $Z(R)$ of zero-divisors in a ring does not possess any obvious algebraic structure; consequently, the study of this set has often involved techniques and ideas from outside algebra. Several recent attempts, among them [**2, 3**] have focused on studying the so-called *zero-divisor graph* Γ_R, whose vertices are the zero-divisors of R, with xy being an edge if and only if $xy = 0$. This object Γ_R is somewhat unwieldy in that it has many symmetries; for example, if $u \in R^*$ is any unit, then $x \mapsto ux$ induces a (graph) automorphism of Γ_R. One way of treating this issue, following an idea of Lauve [**5**], is to work with the *ideal zero-divisor graph* $\mathcal{I}_R$. In effect, one replaces zero-divisors of R by proper ideals with nonzero annihilator; this is the approach adopted by the authors in [**1**]. Such a perspective also has its shortcomings; for instance, it does not adequately detect the phenomenon of there being three distinct proper ideals I, J, K in R with $IJK = 0$, but $IJ \neq 0$, $IK \neq 0$, $JK \neq 0$.

In this paper we adopt a different philosophy, using a new type of homology to study $Z(R)$ and capture the situation described above. Roughly speaking, if we denote by $\mathbf{Z}_n(R)$ the free abelian group generated by the set of $(n+1)$-tuples $(I_0, \dots, I_n)$ of distinct ideals of R such that $I_0 \cdot \cdots \cdot I_n \neq 0$, there are obvious maps $\mathbf{Z}_n(R) \to \mathbf{Z}_{n-1}(R)$ obtained by forgetting one of the factors. This gives $\mathbf{Z}_{.}(R)$ the structure of a semi-simplicial abelian group; hence, we may speak of its associated chain complex $\mathbf{C}_{.}(R)$. Our homology groups $H_*(R)$ are then defined as the homology groups of a certain quotient of $\mathbf{C}_{.}(R)$. The idea behind this construction was sketched by Lauve in [**5**], although the precise definition is due to the authors.

Received by the editors on June 25, 2004, and in revised form on Feb. 9, 2005.