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REFINED ARITHMETIC, GEOMETRIC AND HARMONIC MEAN INEQUALITIES

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Dedicated to Mari Mercer, in loving memory

ABSTRACT. We obtain refinements of the arithmetic, geometric, and harmonic mean inequalities. A main ingredient is Hadamard's inequality. In an application, we obtain a refined version of Ky Fan's inequality.

1. Preliminaries. For $n \geq 2$, let $x_1, x_2, \dots, x_n$ be positive numbers, and let $w_1, w_2, \dots, w_n$ be positive weights: $\sum w_j = 1$. We denote by

$$A = \sum_{j=1}^{n} w_j x_j, \quad G = \prod_{j=1}^{n} x_j^{w_j}, \quad H = \left(\sum_{j=1}^{n} \frac{w_j}{x_j} \right)^{-1},$$

the (weighted) arithmetic, geometric, and harmonic means of the x_j's.

It is well known that

$$H \leq G \leq A,$$

with the inequalities being strict unless all x_j's are equal.

In this paper we obtain various refinements, including upper and lower bounds for $A-G$, $A-H$, A/G and G/H. An important ingredient in our approach is the following.

Hadamard's inequality. *Let f be a concave function on $[a, b]$. Then*

$$\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(t)\, dt \leq f\left(\frac{a+b}{2} \right).$$

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