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REMOVING THE JUMP-KATO'S DECOMPOSITION

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ABSTRACT. A simple proof, using the adjoint operator and Hahn-Banach theorem, is given of Kato's Decomposition which removes the jump at the origin in the nullity (or defect) of a semi-Fredholm operator by subtracting a finite dimensional summand.

Let X be a Banach space over the complex field and let $B(X)$ denote the Banach algebra of bounded linear operators on X. For $T \in B(X)$ set $n(T) = \dim \ker(T)$ and $d(T) = \operatorname{codim} T(X)$. Define the *generalised kernel* $\mathbf{K}(T)$ and the *generalised range* $\mathbf{R}(T)$ of T to be the subspaces

$$\mathbf{K}(T) = \bigcup_1^\infty \ker(T^n), \quad \mathbf{R}(T) = \bigcap_1^\infty T^n(X).$$

Write

$$\Phi_+(X) = \{T \in B(X) : n(T) < \infty \text{ and } T(X) \text{ is closed in } X\},$$
$$\Phi_-(X) = \{T \in B(X) : d(T) < \infty \text{ and } T(X) \text{ is closed in } X\}.$$

$\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ is the set of semi-Fredholm operators in $B(X)$,while $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ is the set of Fredholm operators in $B(X)$. If $T \in \Phi_\pm(X), i(T) = n(T) - d(T)$, a finite or infinite integer, is the index of T. X^* denotes the dual space of X and T^* the adjoint operator of T.

If $T \in \Phi_\pm(X)$, then $\mathbf{R}(T)$ is a closed subspace of T, and if $T_R = T|\mathbf{R}(T)$ denotes the restriction operator, then it is well known [**3**] that $n(T_R) \leq n(T), n(T_R + \lambda) = n(T + \lambda)$ for $\lambda \neq 0, d(T_R) = 0$ and $T_R \in \Phi(\mathbf{R}(T))$. This result is important in that it reduces properties of semi-Fredholm operators to those of Fredholm operators.

If $T \in \Phi_+(X)$ then $\exists \varepsilon > 0$ such that $n(T + \lambda)$ is constant $(\leq n(T))$ for $0 < |\lambda| < \varepsilon$, while if $T \in \Phi_-(X)$ the same is true of $d(T + \lambda)$. Therefore we can define the *jump* of T

$$j(T) = n(T) - n(T + \lambda), \quad 0 < |\lambda| < \varepsilon, \text{ for } T \in \Phi_+(X)$$

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