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ON ALMOST COMMUTING HERMITIAN OPERATORS

STANISLAW J. SZAREK

ABSTRACT. It is an old problem in operator theory whether a pair of norm one compact Hermitian operators with "small" (in norm) commutator can be "well" approximated by a commuting pair of Hermitian operators. We show that, for operators of rank not exceeding n, such approximants exist provided $||[A,B]||/n^{1/2}$ is small. This improves a result of Pearcy and Shields and sheds some new light on the original question and its relationship to a few related ones.

The following is an old question in the "local" operator theory (cf. [**8**]): If two norm one compact Hermitian operators have small commutators, are they close to a commuting pair? More precisely,

(1) *Given $\varepsilon > 0$, does there exist $\delta > 0$ such that, whenever A, B are norm one, compact Hermitian operators on a Hilbert space with $||[A,B]|| \leq \delta$, then one can find* (*compact Hermitian operators*) *A_1, B_1 satisfying $||A_1 - A|| \leq \varepsilon, ||B_1 - B|| \leq \varepsilon$ and $[A_1, B_1] = 0$?*

We are going to refer to (1) as the Main Problem. An equivalent version follows: *If T is a norm one compact operator with "small" selfcommutator $[T^*, T]$, is T "close" to a normal operator?* This one is clearly related to the work of Brown, Douglas, Filmore [**3**] on essentially normal operators. By approximation, questions of the above type reduce to the case of operators acting on finite dimensional spaces (i.e., to matrices) with dimension-free dependence of δ on ε.

Two positive results in the direction of the Main Problem are certainly worth mentioning. First, it was proved by Pearcy and Shields [**7**] that, if just *one* of the operators is assumed to be Hermitian and they

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