ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 2, Spring 1990

ON THE MULTIPLICITY OF $\mathbf{T} \oplus \mathbf{T} \oplus \cdots \oplus \mathbf{T}$

DOMINGO A. HERRERO AND WARREN R. WOGEN

To the memory of our friends and colleagues
Constantin Apostol and Doug McMahon

1. Introduction. Let $\mathcal{L}(\mathcal{X})$ denote the algebra of all (bounded linear) operators on a complex Banach space $\mathcal{X}$. The *multiplicity* of $T \in \mathcal{L}(\mathcal{X})$ is the cardinal number defined by

$$\mu(T) = \min_{\Gamma \subset \mathcal{X}} \{\operatorname{card} \Gamma : \mathcal{X} = \bigvee \{T^k y : y \in \Gamma, k = 0, 1, 2, \dots\}\},$$

where $\bigvee \mathcal{R}$ denotes the closed linear span of the vectors in $\mathcal{R}$.

If $\mu(T)$ is finite or denumerable, then $\mathcal{X}$ is necessarily separable. Throughout this note we shall always assume that $\mathcal{X}$ is *separable* and *infinite dimensional*.

If $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y})$, then $A \oplus B$ denotes the direct sum of A and B acting in the usual fashion on the hilbertian direct sum $\mathcal{X} \oplus \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$. It is an easy exercise to check that $\max[\mu(A), \mu(B)] \leq \mu(A \oplus B) \leq \mu(A) + \mu(B)$.

Let $T \in \mathcal{L}(\mathcal{X})$; for each $n \geq 1$, let $T^{(n)}$ denote the direct sum of n copies of T acting in the usual fashion of the direct sum $\mathcal{X}^{(n)}$ of n copies of $\mathcal{X}$. It readily follows from the previous observations that

$$\begin{aligned}\max[\mu(T^{(m)}), \mu(T^{(n)})] = \mu(T^{(\max[m,n])}) &\leq \mu(T^{(m+n)}) \\ &\leq \mu(T^{(m)}) + \mu(T^{(n)}), m, n \geq 1.\end{aligned}$$

For which sequences $\{\mu_n\}_{n=1}^{\infty}$ of natural numbers satisfying the conditions $\mu_{\max[m,n]} \leq \mu_{m+n} \leq \mu_m + \mu_n, m, n \geq 1$, does there exist a Banach space operator T such that $\mu(T^{(n)}) = \mu_n$ for all $n = 1, 2, \dots$?

This research was partially supported by Grants of the National Science Foundation.