

LOCAL SEMIGROUPS IN LIE GROUPS AND LOCALLY REACHABLE SETS

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ABSTRACT. In this paper the relationship between locally reachable sets for a fixed set of controls in the Lie algebra of a Lie group and the local semigroups generated by the corresponding one-parameter semigroup is considered. It is convenient to carry out the investigation in the Lie algebra itself, and the appropriate machinery for doing this is first developed. It is shown that local semigroups contain locally reachable sets. A general criterion (rerouting) is developed for the converse inclusion, and it is shown that if the set of controls is contained in a proper cone or is a Lie wedge (i.e., the tangent of a local semigroup), then it is the case that locally reachable sets contain local semigroups.

S. Lie's Fundamental Theorem states that there is a bijection between the subalgebras of a given finite dimensional real Lie algebra L and local subgroups of a fixed local group having L as Lie algebra. In recent years there has been considerable interest in studying (local) subsemigroups of (local) Lie groups [3, 4, 6, 7-9, 10, 11], partly because of their relevance in geometric control theory, partly because of their occurrence in the theory of symmetric spaces and "causal" semigroups, and partly in order to complete S. Lie's original program.

For the local study of subsemigroups of a Lie group it is convenient to stay inside the given Lie algebra L and to fix a convex symmetric open neighborhood on which the Campbell-Hausdorff-Baker multiplication $(x, y) \rightarrow x * y = x + y + [x, y]/2 + \cdots$ is defined as a function $B \times B \rightarrow L$ through the absolute convergence of the Campbell-Hausdorff-Baker series. Such a neighborhood we will call a *CHB-neighborhood* (short for *Campbell-Hausdorff-Baker-neighborhood*). (This approach is no restriction in considering local theory since the exponential mapping is a local analytic isomorphism from B with the CHB-multiplication

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