ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 1, Winter 1991

SUBORDINATION OF POLYNOMIALS

ANTONIO CÓRDOVA AND STEPHAN RUSCHEWEYH

ABSTRACT. We prove a general sufficient condition for polynomials to be subordinate to certain analytic functions. This generalizes and unifies other known conditions.

1. Introduction. Let $\mathcal{A}$ denote the set of analytic functions in the unit disk $\mathbf{D}$ of the complex plane $\mathbf{C}$, and let $\mathcal{P}_n$ be the set of polynomials of degree $\leq n$. If $f, g \in \mathcal{A}$, then f is called *subordinate* to g (denoted by $f \prec g$) if $f = g \circ \omega$, where $\omega \in \mathcal{A}$ satisfies $|\omega(z)| \leq |z|$ in $\mathbf{D}$. The concept of subordination turned out to be very useful for many applications in function theory, and, therefore, it is of interest to obtain simple, sufficient criteria which guarantee subordination. The most elementary one of such criteria works if g happens to be univalent in $\mathbf{D}$. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbf{D}) \subset \mathbf{g}(\mathbf{D})$. Less immediate, but nevertheless easy to prove, is the following known special result.

Theorem A. *Let $P \in \mathcal{P}_n$, $P(0) = 1$ and $0 \notin P(\mathbf{D})$. Then $P \prec Q_n := (1 - z)^n$.*

Note that here already the omission of *one single boundary point* of $Q_n(\mathbf{D})$, namely, $Q_n(1) = 0$, guarantees subordination. Theorem A is fairly surprising, and it seems to represent the only known case, so far, of this kind of subordination criterion. In connection with our work [**1, 2**] on the "maximal range" problem for polynomials, we were led to conjecture that the following might be true.

This work has been supported by the Fondo Nacional Científico y Tecnológico (FONDECYT), Grant 249/87, by the Universidad F. Santa María (Grant 881206), and by the German Academic Exchange Service (DAAD).
Received by the editors on August 18, 1988, and in revised form on December 15, 1989.