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CLOSED SUBALGEBRAS OF THE BANACH ALGEBRA OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS ON AN INTERVAL

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ABSTRACT. Several classes of closed subalgebras of $D^1(I)$ are studied in this paper. A number of results on singly-generated subalgebras are given, including the result that such subalgebras are regular (in the sense of Shilov) if and only if the range of the generator does not separate the plane. Other conditions sufficient for closed subalgebras to be regular are also given. For instance, closed separating subalgebras are shown to be regular and singly-generated. The paper closes with a characterization of those closed separating subalgebras over which $D^1(I)$ is integral.

Introduction. In this paper, we are concerned with closed subalgebras of the Banach algebra $D^1(I)$ of continuously differentiable complex-valued functions on a closed interval $I = [a, b]$ of real numbers. The norm $\|\cdot\|_1$ on $D^1(I)$ is defined by $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$, $f \in D^1(I)$, where $\|\cdot\|_\infty$ is the sup norm on I. We shall always assume that our subalgebras contain the constant functions.

Sections one and two are concerned primarily with the singly-generated closed subalgebras of $D^1(I)$. For $f \in D^1(I)$ we use A_f to denote the closed subalgebra generated by f and let S_f be its set of critical points. Section 1 is preliminary in nature, and contains definitions and technical results used later in the paper. In Section 2, we are interested in identifying the functions $g \in D^1(I)$ which must belong to A_f. Obviously, any characterization of such functions must involve the derivative of g. For the case where the range $f(I)$ of f does not separate the plane, we show (Theorem 2.1) that $g \in A_f$ if and only if $f(x) = f(y)$ implies $g(x) = g(y)$ and $g'(x)f'(y) = g'(y)f'(x)$, and $f'(x) = 0$ implies $g'(x) = 0$. Also in Section 2, we refine our results for the case where f is real-valued.

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