ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 4, Fall 1991

SOME REMARKS ON REPRODUCING KERNEL KREIN SPACES

DANIEL ALPAY

ABSTRACT. The one-to-one correspondence between positive functions and reproducing kernel Hilbert spaces was extended by L. Schwartz to a (onto, but not one-to-one) correspondence between difference of positive functions and reproducing kernel Krein spaces. After discussing this result, we prove that a matrix valued function $K(z,\omega)$ symmetric and jointly analytic in z and $\bar{\omega}$ in a neighborhood of the origin is the reproducing kernel of a reproducing kernel Krein space. We conclude with an example showing that such a function can be the reproducing kernel of two different Krein spaces.

1. Introduction. In this paper we study some points in the theory of reproducing kernel Krein spaces, namely existence theorems and a nonuniqueness counterexample. We begin with a brief review of reproducing kernel spaces, which helps setting the framework and gives some motivation.

Let $K(z,\omega)$ be a $\mathbf{C}_{n\times n}$-valued function for z and ω in some set Ω, and let V be a vector space of $\mathbf{C}_n$-valued functions defined on Ω, endowed with some hermitian form $[\ ,\ ]_V$. The function K is a reproducing kernel for V if the following two conditions hold:

(a) for any ω in Ω and c in $\mathbf{C}_n$, the function $K_\omega c : z \to K(z,\omega)c$ belongs to V.

(b) for any f in V, ω in Ω and c in $\mathbf{C}_n$,

$$[f, K_\omega c]_V = c^* f(\omega) \tag{1.1}$$

($\mathbf{C}_{n\times l}$ denotes the vector space n-rows l-columns matrices with complex entries, $\mathbf{C}_{n\times 1}$ is written $\mathbf{C}_n$ and A^* denotes the adjoint of the matrix A.)

It is easy to check that $(V, [\ ,\ ]_V)$ has at most one reproducing kernel, which is moreover a symmetric function (also called hermitian)

$$K(z,\omega) = (K(\omega,z))^*. \tag{1.2}$$

Received by the editors on October 4, 1988.