ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 1, Winter 1992

ON A FAMILY OF CONVEX POLYNOMIALS

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Consider the nth partial sum of the series $e^{1+z} = \sum_{k=0}^{\infty}((1+z)^k/k!)$. Set $P_n(z) = \sum_{k=0}^{n}((1+z)^k/k!)$ and note that $P_{n-1}(z) = P_n'(z)$. We wish to show that $P_n(D)$ is convex where $D = \{|z| < 1\}$, $n \geq 1$. The proof is by induction. Clearly $P_1(D)$ is convex. Also, $P_2(z) = (5/2) + 2z + (z^2/2)$ and it is easy to see that $P_2(D)$ is convex. That is,

$$\mathrm{Re}\left[\frac{zP_2''}{P_2'} + 1\right] = \mathrm{Re}\left[\frac{2+2z}{2+z}\right] > 0$$

when $|z| < 1$.

Suppose it is known $P_k(D)$ is convex for $k < n$ where $n \geq 3$. Because of the convexity and the fact that all the coefficients are positive, $\mathrm{Re}\,(P_n'(z)) = \mathrm{Re}\,(P_{n-1}(z)) \geq P_{n-1}(-1) = 1$ so that $|P_n'(z)| \geq 1$, $|z| \leq 1$.

Thus, we have

$$\begin{aligned} zP_n''(z) + P_n'(z) &= P_{n-1}(z) + zP_{n-2}(z) \\ &= P_{n-1}(z) + z\left[P_{n-1}(z) - \frac{(1+z)^{n-1}}{(n-1)!}\right] \\ &= (1+z)P_{n-1}(z) - \frac{z(1+z)^{n-1}}{(n-1)!}. \end{aligned}$$

Since the minimum value of a harmonic function occurs on the boundary, we set $z = e^{i\theta}$ and see that

$$\begin{aligned} \mathrm{Re}\left[1 + z - \frac{z(1+z)^{n-1}}{(n-1)!P_n'(z)}\right] &\geq 1 + \cos\theta - \frac{|1+z|^{n-1}}{(n-1)!} \\ &\geq (1+\cos\theta) - \frac{(1+\cos\theta)2^{n-2}}{(n-1)!} \\ &= (1+\cos\theta)\left(1 - \frac{2^{n-2}}{(n-1)!}\right) \\ &\geq 0 \end{aligned}$$

Received by the editors on December 17, 1988.
1980 *AMS Subject Classification.* 30C10, 30C45, 30C50.