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FUCHS' PROBLEM 43

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What is the relationship between the abelian groups A and C, if $\mathrm{Ext}\,(A,B) \cong \mathrm{Ext}\,(C,B)$ for every abelian group B? This is problem 43 in [**3**]. In this note we give a complete solution to this problem when A, B and C are torsion-free abelian groups of finite rank. Our approach is to show that numerical invariants considered in [**5**] actually characterize the reduced finite rank torsion-free groups up to quasi-isomorphism.

This paper is essentially self-contained; however, the reader may wish to refer to [**1, 3**, and **4**]. For $B \leq A$, we say that B is a quasi-summand of A if for some $n \neq 0$ and $A' \leq A$, $nA \leq B \oplus A' \leq A$, and A is called strongly indecomposable in case A has no nontrivial quasi-summands. If $C \cong B$ and $nA \leq B \leq A$, then A and C are called quasi-isomorphic. As usual, set $QA = Q \otimes_Z A$ and regard $A \leq QA$.

Let $S_A(C)$ be the subgroup of C generated by $f(A)$ for all $f \in \mathrm{Hom}\,(A,C)$. A subgroup B of C will be said to be full in C if $\langle B \rangle_* = C$ where $\langle B \rangle_*$ denotes the pure subgroup of C generated by B.

Below all groups are torsion-free. The quasi-endomorphism ring of A is $QE(A)$ where $E(A)$ is the endomorphism ring of A. By the well-known result of J. Reid, $QE(A)$ is left Artinian if and only if A is quasi-isomorphic to a finite direct sum $A_1 \oplus \cdots \oplus A_n$ with each A_i strongly indecomposable. Moreover, if $\alpha \in QE(A_i)$, then α is invertible or α is nilpotent [**7**]. The proof of the main theorem will rest upon the

Lemma. *Let A and C be torsion-free groups with left Artinian quasi-endomorphism rings. If $S_A(C)$ is full in C and $S_C(A)$ is full in A, then A and C have an isomorphic nonzero quasi-summand.*

Proof. Let $E = E(A)$ and let R denote the nilradical of E. For $J =$ Jacobson radical of QE, $R = J \cap E$, and since QE is left Artinian, J (hence R) is nilpotent. Call $N = \langle RA \rangle_*$ which is the pure subgroup

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