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# $\ell^{p,\infty}$ HAS A COMPLEMENTED SUBSPACE ISOMORPHIC TO $\ell^2$

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ABSTRACT. We prove the assertion claimed in the title.

The weak $L^p$ spaces play an important role in interpolation theory as well as harmonic analysis. In this paper, we concentrate on their geometry as Banach spaces. More precisely, we show that the sequential weak $L^p$ spaces have complemented subspaces isomorphic to $\ell^2$. We remark that the corresponding assertion for non-atomic weak $L^p$ spaces is also true. For, in this case, it is easy to see that the Rademacher functions span a complemented subspace isomorphic to $\ell^2$. The idea of the proof in the atomic case is to create a similar situation in the $\ell^{p,\infty}$ spaces.

We start by recalling some standard definitions. Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space. For $1 < p < \infty$, the weak $L^p$ space $L(p, \infty, \mu)$ is the space of all $\Sigma$-measurable functions $f$ such that $\{\omega : |f(\omega)| > 0\}$ is $\sigma$-finite and $||f|| \equiv \sup_B \int_B |f|\, d\mu/\mu(B)^{1-1/p} < \infty$, where the supremum is taken over all measurable sets $B$ with $0 < \mu(B) < \infty$. In case $(\Omega, \Sigma, \mu) = [0,1]$ endowed with Lebesgue measure or $\mathbf{N}$ with the counting measure, we use the notation $L^{p,\infty}[0,1]$ and $\ell^{p,\infty}$, respectively. For an real valued function $f$ defined on $(\Omega, \Sigma, \mu)$, we denote by $f^*$ the decreasing rearrangement of $|f|$ [**1**], similarly for $(a_n^*)$, where $(a_n)$ is a sequence of real numbers. It is well known that $L^{p,\infty}[0,1]$ is naturally isomorphic to the dual of $L^{q,1}[0,1]$, $1/p + 1/q = 1$, where $L^{q,1}[0,1]$ denotes the space of all measurable functions $f$ such that $||f|| = \int_0^1 t^{-1/p} f^*(t)\, dt < \infty$. Furthermore, $L(p, \infty, \mu)$ satisfies an upper $p$-estimate [**2**]. For further details on the weak $L^p$ spaces, we refer to [**2, 1**].

Fix $1 < p < \infty$, let $(e_i)$ denote the coordinate unit vectors in $\ell^{p,\infty}$, and let $F = [e_i]$.

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