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THE DISTRIBUTION OF RELATIVELY r-PRIME INTEGERS IN RESIDUE CLASSES

J.E. NYMANN

ABSTRACT. If 1 is the only r-th power which is a divisor of $m_1, m_2, \dots, m_k$, then $m_1, m_2, \dots, m_k$ are said to be relatively r-prime. If $\bar{a} = \langle a_1, a_2, \dots, a_k \rangle$ is a k-tuple of nonnegative integers, h is a positive integer and x is a positive real number, let $Q(x; \bar{a}, h, r, k)$ denote the number of k-tuples of positive integers $\langle m_1, m_2, \dots, m_k \rangle$ for which $1 \le m_i \le x$, $m_i \equiv a_i \pmod{h}$, $i = 1, 2, \dots, k$ and $m_1, m_2, \dots, m_k$ are relatively r-prime. An asymptotic formula with 0-estimate for $Q(x; \bar{a}, h, r, k)$ is determined. Special cases of this estimate give earlier estimates for relatively prime integers and r-free integers.

1. Introduction. For $m_1, m_2, \dots, m_k$ integers and r a positive integer we write $(m_1, m_2, \dots, m_k)_r = d^r$ if d is the largest integer for which $d^r \mid m_i (i = 1, 2, \dots, k)$. If $(m_1, m_2, \dots, m_k)_r = 1$, we say $m_1, m_2, \dots, m_k$ are relatively r-prime. Note that in the case $k = 1$, $(m)_r = 1$ means m is r-free. For $a_1, a_2, \dots, a_k$ nonnegative integers, $\bar{a}$ will denote the k-tuple $\langle a_1, a_2, \dots, a_k \rangle$. For h a positive integer and x a positive real number, $Q(x; \bar{a}, h, r, k)$ will denote the number of k-tuples of positive integers $\langle m_1, m_2, \dots, m_k \rangle$ for which $1 \le m_i \le x$, $m_i \equiv a_i \pmod{h}$, $i = 1, 2, \dots, k$ and $(m_1, m_2, \dots, m_k)_r = 1$.

Letting $g = (a_1, a_2, \dots, a_k)$, it is not difficult to see that if $(g, h)_r \ne 1$, then $Q(x; \bar{a}, h, r, k) = 0$ for all x. Section 3 of this paper is devoted to obtaining an asymptotic formula with 0-estimate for $Q(x; \bar{a}, h, r, k)$ in the case $(g, h)_r \ne 1$. The remaining sections are devoted to showing that special cases of this result give earlier results on the distribution of relatively prime integers and r-free integers, and examining questions of equidistribution of relatively r-prime k-tuples in the (admissible) k-tuples of residue classes $\pmod{h}$.

2. Preliminaries. A divisor d of n is said to be a unitary divisor if $(d, n/d) = 1$. We write $(a, n)_* = d$ if d is the largest unitary divisor of n which divides a. $\phi^*(n)$ denotes the number of positive integers $a \le n$ for which $(a, n)_* = 1$. Noting first that ϕ^* is multiplicative, it is not