ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Fall 1992

NEUTRAL GEOMETRY AND THE GAUSS-BONNET THEOREM FOR TWO-DIMENSIONAL PSEUDO-RIEMANNIAN MANIFOLDS

PETER R. LAW

1. Introduction. The Gauss-Bonnet theorem was first extended to pseudo-Riemannian manifolds by Avez [**1**] and Chern [**3**]. These authors produced a global Gauss-Bonnet theorem. For example, Chern [**3**] considers oriented pseudo-Riemannian vector bundles of even rank over compact manifolds and interprets the Gauss-Bonnet formula as the assertion that the relevant curvature form (that which appears as the integrand in the Gauss-Bonnet formula) equals the Euler class of the bundle. This is now the standard abstract formulation of the generalized Gauss-Bonnet theorem, though usually stated only for the Riemannian case (cf., e.g., Milnor and Stasheff [**7**]). For the tangent bundle of a compact, oriented, pseudo-Riemannian manifold, this statement reduces to the usual Gauss-Bonnet result.

The obvious elegance of this global Gauss-Bonnet-Chern theorem does not preclude interest in a pseudo-Riemannian version of the classical Gauss-Bonnet formula for a two-dimensional domain D with piecewise smooth boundary Γ:

$$\int_D K\,dV + \int_\Gamma k_g\,ds + \sum \theta_{\text{exterior}} = 2\pi \tag{1.1}$$

where K is the Gaussian curvature of some metric on D, k_g the geodesic curvature, and θ_{exterior} the exterior angle at a nonsmooth point of Γ. It is fairly straightforward to carry over the differential-geometric aspects of a proof of this result to the pseudo-Riemannian context. The only two-dimensional indefinite signature is Lorentzian, and the essential difference that occur between the Riemannian and Lorentzian versions of the Gauss-Bonnet formula arise from the differences between the corresponding orientation-preserving isometry groups $\mathbf{SO}\,(\mathbf{2})$ and $\mathbf{SO}\,(\mathbf{1},\mathbf{1})$. In particular, these groups essentially determine the relevant notion of angle and hence θ_{exterior}.

Received by the editors on January 9, 1990.