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DO SUBSPACES HAVE DISTINGUISHED BASES?

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While trying to develop a computer program to calculate resolutions for modules over path algebras, the second author conjectured the existence of an abstract version of the Gram-Schmidt process. Given a basis for a vector space, there seemed to be an "algorithmically preferred" basis for each subspace. Although this idea is quite simple-minded, it does not appear explicitly in any of the standard treatments of elementary linear algebra. On the other hand, mathematics teachers will recognize our observation as a concrete description of what we have all noticed and tried to explain when teaching Gaussian elimination. In clarifying the obvious we provide some insights into the construction of Gröbner bases, a fundamental tool in computational algebra.

We wish to take advantage of the ordering in an ordered basis for a vector space. Sometimes a concrete space comes equipped with a natural ordered basis and, sometimes, as we shall see in an application to diagonalizability, the ordering can be quite arbitrary.

Example 1. Let K be a field and let $V = K^n$ be the vector space of n-tuples with coordinates from K. The standard basis $e_1, \dots, e_n$ has a standard well-ordering, namely $e_1 < e_2 < \cdots < e_n$. In our discussion of row echelon form we refer to the *reverse* ordering on the standard basis: $e_n < e_{n-1} < \cdots < e_1$.

We introduce definitions and notations which will be used in the remainder of the paper. Let V be a vector space over a field K with a given basis B which is well-ordered by $<$. Each $v \in V$ can be written in a unique way as a linear combination of members of B; if $b \in B$ and its coefficient in this linear combination is nonzero, we will say that b *occurs in* v. The maximal $b \in B$ (by the ordering of B) which occurs in v is called the *tip of* v. If X is a nonempty subset of V, then $\mathrm{TIP}(X)$ will consist of all basis elements in B which occur as the tip of

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