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QUASI-DECOMPOSITION OF ABELIAN GROUPS AND BAER'S LEMMA

ULRICH ALBRECHT

1. Introduction. In this paper we consider an abelian group A. The *class of A-projective groups* (*of finite A-rank*) is obtained by closing $\{A\}$ under (finite) direct sums and direct summands. The *A-socle of an abelian group G*, which is denoted by $S_A(G)$, is the subgroup of G which is generated by $\{\phi(A)|\phi \in \text{Hom}\,(A,G)\}$. Finally, the group A is *self-small* if, for all index-sets I and all $\alpha \in \text{Hom}\,(A, \oplus_I A)$, there is a finite subset J of I with $\alpha(A) \subseteq \oplus_j A$. Clearly, torsion-free abelian groups of finite rank are self-small. Other examples of self-small abelian groups can be found in [**7**].

The group A has the (*finite*) *Baer-splitting property* if every exact sequence $0 \to B \stackrel{\alpha}{\to} G \stackrel{\beta}{\to} P \to 0$ such that $\alpha(B) + S_A(G) = G$ and P is A-projective (of finite A-rank) splits. Baer verified in [**8**] that every subgroup of the rational numbers has the Baer-splitting property. In [**3**, Theorem 2.1 and Corollary 2.2], a complete characterization of the self-small abelian groups A having the (finite) Baer-splitting property was obtained which extends Arnold's and Lady's results of [**6**].

Unfortunately, a splitting result like Baer's Lemma often has limited applications in the discussion of torsion-free groups of finite rank since the splitting of a short exact sequence of these groups occurs less frequently than its quasi-splitting. Because of this, we introduce the following weaker, but perhaps more useful version of the Baer-splitting property which is based on the idea of quasi-isomorphism introduced by Jonsson in the 1950s ([**10,11**]): A torsion-free abelian group A has the (*finite*) *quasi-Baer-splitting property* if every exact sequence $0 \to B \stackrel{\alpha}{\to} C \stackrel{\beta}{\to} G \to 0$, in which G is isomorphic to a torsion-free quasi-summand of an A-projective group (of finite A-rank), and C and $\alpha(B) + S_A(C)$ are quasi-equal, quasi-splits. Theorem 2.3 and Corollary 2.4 give a complete characterization of the self-small abelian groups A having the (finite) quasi-Baer-splitting property in terms of the $E(A)$-module structure of A where the symbol $E(A)$ denotes the

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