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DEGREES OF CLOSED CURVES IN THE PLANE

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ABSTRACT. In the present article we extend the notion of degree from regular closed curves to closed locally one-to-one curves and prove that the extended notion has analogous properties. In particular, a natural generalization of Whitney-Graustein's theorem is still true. A proof of a mean value theorem for nonstop curves is given using only the elementary ideas of this paper.

1. Introduction. Let us first recall some important definitions.

A curve $C : I \to \mathbf{R}^2$ is *regular* if it is continuously differentiable and if $C'(t) \neq 0$ for all $t \in I$.

The map $H : I \times I \to \mathbf{R}^2$ is a *regular homotopy* if the curve $H_u(t) = H(u,t)$ is regular for each u and if both H_u and its derivative vary continuously with u. If H is a regular homotopy, then H_0 and H_1 are said to be regularly homotopic.

If $C : I \to \mathbf{R}^2$ is a (continuous) curve such that $C(t) \neq 0$ for all $t \in I$, then the *winding number* $W(C)$ of C around 0 is defined as follows. Identify $\mathbf{R}^2$ with the complex plane, and write C as $C(t) = r(t)e^{2\pi i a(t)}$ where both r and a are continuous functions and r is positive. Let $W(C)$ be the difference $a(1) - a(0)$. If C is a closed curve $W(C)$ is clearly an integer. W is also homotopy invariant in the following sense: if two curves $C_1, C_2 : I \to \mathbf{R}^2$ are homotopic by a homotopy $H : I \times I \to \mathbf{R}^2 - \{0\}$ such that H_u, defined by $H_u(t) = H(u,t)$, is a closed curve for all $u \in I$, then $W(C_1) = W(C_2)$. The winding number of a curve counts the algebraic number of times the curve goes around the origin. If C is a closed curve it follows from the definition of the winding number that the vector $C(t)$ points in every direction for at least $|W(C)|$ different values of t. A very readable discussion of the winding number is given in [**1**].

If C Is a regular closed curve, then C' is a closed curve in $\mathbf{R}^2$ missing the origin. Therefore, $W(C')$ can be defined. $D(C) = W(C')$ is called

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