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COVARIANT COMPLETELY POSITIVE MAPS AND LIFTINGS

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ABSTRACT. Covariant completely positive maps related to discrete unital C^*-dynamical systems are studied. We consider their application to the completely positive lifting problem for homomorphisms of certain reduced discrete group C^*-algebras into the Calkin algebra.

Let $\alpha : G \to \operatorname{Aut}(A)$ be an action of a discrete group G on a C^*-algebra A. Given a unitary representation $u : G \to \mathcal{U}(B)$ of G into the unitary group of a unital C^*-algebra B, a linear map $\varphi : A \to B$ is called *u-covariant* if $\varphi(\alpha_g(a)) = u_g\varphi(a)u_g^*$ for all $a \in A$ and $g \in G$. In this situation the triple (A, G, α) is called a discrete C^*-dynamical system, and φ will be referred to as a *(discrete) covariant map of* (A, G, α).

The purpose of this note is to explore the natural relation of discrete covariant completely positive maps to crossed products and to show their application to a certain lifting problem. We shall show that a discrete covariant completely positive map $\varphi : A \to B$ extends to a completely positive map on the crossed product $A \times_\alpha G$. This is done using the analog of the covariant version of Stinespring theorem [**8**] for bounded operators on Hilbert modules. Our main result states that, given a discrete unital C^*-dynamical system (A, G, α) such that $A \times_\alpha G$ nuclear and $A \times_\alpha G = A \times_{\alpha r} G$, a homomorphism $\sigma : C_r^*(G) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ has a completely positive lifting to $\mathcal{B}(\mathcal{H})$ precisely when there exists a covariant completely positive map $\varphi : A \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ with respect to the unitary representation of G in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, determined by σ. As an illustration, it follows that the invertible elements of the semigroup Ext of the Choi's algebra $C_r^*(\mathbf{Z}_2 * \mathbf{Z}_3)$ are determined by those monomorphisms $\sigma : C_r^*(\mathbf{Z}_2 * \mathbf{Z}_3) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ which can be extended to completely positive maps on the Cuntz algebra $\mathcal{O}_2$.

Recall that each covariant representation (π, u) of (A, G, α) on a Hilbert space canonically induces a representation $\pi \times u$ of the involutive

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