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THE RIESZ INTEGRAL AND AN $L^p - L^q$ ESTIMATE FOR THE CAUCHY PROBLEM OF THE WAVE OPERATOR

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ABSTRACT. In 1949, M. Riesz [**3**] generalized the Riemann-Liouville integral of one-variable to high dimensional Euclidean spaces and obtained a powerful method now known as the Riesz integral for studying wave operators. In this paper we apply the Riesz integral to get the global space-time estimate

$$||u||_q \leq C\{||w||_p + t^{(1-n)/(n+1)}(||g||_p + ||\nabla f||_p)\}$$

where $1/q = 1/p - 2/(n+1)$, $1/p + 1/q = 1$, and u is the solution of the Cauchy problem $\Box u(x,t) = w(x,t)$ in $\mathbf{R}^{n+1}_+$, $u(x,0) = f(x)$, and $\partial_t u(x,0) = g(x)$.

1. The Riesz distribution. For $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, we denote $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Let $\mathbf{R}^{n+1} = \{(x,t) : x \in \mathbf{R}^n, t \in \mathbf{R}\}$, and define

$$\rho^\lambda = \begin{cases} (t^2 - |x|^2)^{\lambda/2} & \text{if } t \geq |x| \\ 0 & \text{otherwise.} \end{cases}$$

For $\operatorname{Re}\lambda > -2$, ρ^λ is a locally integrable function on $\mathbf{R}^{n+1}$ and so defines a distribution

$$\langle \rho^\lambda, \phi \rangle = \int_{\mathbf{R}^{n+1}} \rho^\lambda \phi(x,t)\, dx\, dt$$

for $\phi \in \mathcal{D}(\mathbf{R}^{n+1})$. In spherical coordinates, the above integral can be written as

$$\langle \rho^\lambda, \phi \rangle = \int_0^\infty \int_0^t (t^2 - r^2)^{\lambda/2} r^{n-1} \bar{\phi}(r,t)\, dr\, dt$$

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