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ON A CONJECTURE OF R.L. GRAHAM

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1. Introduction. Let a and b be two positive integers. We shall call the ratio $a/(a,b)$ the *reduced ratio* of a and b. In 1970, R.L. Graham [**5**] conjectured that *the maximum of the reduced ratios of pairs of integers in a finite set of positive integers is at least the cardinality of the set.*

Let $M(n)$ be the least common multiple of $1, 2, \dots, n$. We shall call $\mathcal{S}$ a *standard set* if either

$$\mathcal{S} = \{1, 2, \dots, n\}, \qquad S = \{M(n)/1, M(n)/2, \dots, M(n)/n\},$$

or

$$\mathcal{S} = \{2, 3, 4, 6\}.$$

We say a finite set of positive integers is *primitive* if the greatest common divisor of the members of the set is 1. It is easy to see that if Graham's conjecture is valid for any primitive set, then it is valid in general. One may further conjecture (see [**8**]), that *if the maximum of the reduced ratios of pairs of integers in a primitive set is at most its cardinal number, then the set is standard.* We call this the strong Graham conjecture. It is clear that the strong Graham conjecture implies Graham's conjecture.

Graham's conjecture has gotten much attention since it was proposed and there are many partial results. See [**4**] for a survey up to 1980.

Szegedy [**7**] and Zaharescu [**9**] independently proved Graham's conjecture for all sufficiently large cardinalities. In fact, Szegedy established the strong Graham conjecture for such cardinalities. Both Szegedy's and Zaharescu's proofs rely on 'Hoheisel type' results in prime number theory. Namely, they use the theorem that there is some constant $c > 0$, such that the number of primes in the interval $[x, x+x^{1-c}]$ is asymptotically $x^{1-c}/\log x$ as $x \to \infty$. This is a deep result in analytic number theory, and it would be very difficult to use the Szegedy or Zaharescu proofs to give an explicit bound for "sufficiently large." To be sure,

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