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ON THE EXISTENCE OF TANGENT HYPERPLANES TO FULL SUBLATTICES OF EUCLIDEAN SPACE

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ABSTRACT. Let L be a full sublattice of Euclidean n-space. We study those points in the boundary of L where L admits a tangent hyperplane. The main result states that this collection of points is dense in the boundary of L. This theorem is a generalization of the well-known fact that monotone increasing real-valued functions are differentiable almost everywhere.

1. Introduction. A standard result in analysis states that monotone increasing real-valued functions are differentiable almost everywhere. In other words, if $f : [0,1] \to \mathbf{R}$ is a monotone (upper semicontinuous) function, and if $L = \{(x,y) \in [0,1]^2 : y \leq f(x)\}$ is the subgraph of f, then the set of points where we can assure the existence of a tangent line to L is dense in the boundary of L. In this note we will extend this result to full sublattices: A sublattice $L \subseteq \mathbf{R}^n$ is called *full*, provided that the interior L° of L is connected and dense in L. Full sublattices of $\mathbf{R}^n$ were first introduced and studied in greater detail in [**2**] and [**3**]. If L is such a full sublattice, then the points p in the boundary of L where L admits a tangent hyperplane is dense in the boundary ∂L of L. Such a point $p \in \partial L$ will be called a $\mathcal{C}_1$-point. The property of being a $\mathcal{C}_1$-point is not an intrinsic property of the point $p \in \partial L$; it rather depends on the particular imbedding of L into $\mathbf{R}^n$. On the other hand, there are certain points $p \in \partial L$ that do not admit a tangent plane under any imbedding of L into $\mathbf{R}^n$.

Another related result is S. Mazur's theorem [**5**] which states that a closed convex set with dense interior in a separable Banach space has a dense set of points of $\mathcal{C}_1$-points in the boundary. From a point of view of order theory, convex sets typically stand at the opposite side of distributivity. So one might hope that there is a generalization of Mazur's result to abstract convex structures along the lines studied by

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