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A REVERSE ISOPERIMETRIC INEQUALITY, STABILITY AND EXTREMAL THEOREMS FOR PLANE CURVES WITH BOUNDED CURVATURE

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0. Introduction. In this note we discuss some elementary theorems about the relation between area and length of closed embedded plane curves with bounded curvature. Our main result (see Theorem 4.1) solves the extremal problem of which domain has *largest* boundary length among embedded disks in the plane whose boundary curvatures are uniformly bounded and whose area is fixed and sufficiently small.

Reverse Isoperimetric Inequality. *If M is an embedded closed disk in the plane $\mathbf{R}^2$ whose boundary curvature satisfies $|\kappa| \leq 1$ and with area $A \leq \pi + 2\sqrt{3}$ then the length of ∂M is bounded by*

$$\frac{L - 2\pi}{4} \leq \operatorname{Arcsin}\left(\frac{A - \pi}{4}\right).$$

If equality holds then M is congruent to a peanut-shaped domain as in Figure 1.

This gives an estimate in the reverse direction to the classical isoperimetric inequality. There is also a threshold phenomenon: if the area is larger than $\pi + 2\sqrt{3}$ then there is no upper bound for the length of ∂M. This is the area of the pinched peanut domain $P_{\sqrt{3}}$. Examples can be found by breaking a thin peanut and connecting the ends with a long narrow strip. In fact, the set of possible points (A, L) for embedded disks whose boundary satisfies $|\kappa| \leq 1$ is further restricted (Theorem 4.1). There is a suggestive physical interpretation of the equivalent dual problem, where the length is fixed and the minimal area disk is sought. One may imagine the cross section of a hose in which the inside pressure is smaller than the outside. If the hose has limited flexibility, modelled by a uniform bound on the curvature, then the equilibrium section is again the peanut shape.

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