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SPECTRUM WHERE THE BOUNDARY OF THE NUMERICAL RANGE IS NOT ROUND

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ABSTRACT. For a bounded linear operator A on a complex Hilbert space, we prove that the boundary points of the numerical range $W(A)$ with infinite curvature of the convex boundary curve are included in the spectrum of both A and A^*. If, additionally, $W(A)$ is closed, then the 'non-round' boundary points are eigenvalues of A and A^*.

The numerical range $W(A)$ of the operator A is defined as the set of complex numbers (Au, u) where u runs through the vectors of norm 1. The basic fact concerning numerical range is the Toeplitz-Hausdorff theorem which states that the numerical range of a bounded linear operator on a Hilbert space $\mathcal{H}$ is convex [**2**]. The closure $\overline{W(A)}$ of the numerical range contains the spectrum of A, is convex too and is compact because of boundedness of the operator A. The boundary of $W(A)$ is a Jordan curve and will be called $C(A)$. For some related material on the numerical range of operators, see [**3, 4**].

Convex compact sets have enough extreme points and we would like to ask whether extreme points of $\overline{W(A)}$ belong to the spectrum. The example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

on a 2-dimensional Hilbert space shows that this is not necessarily so; the matrix is nilpotent, has spectrum $\{0\}$ and numerical range equal to the closed disk with center 0 and radius $1/2$. On the other hand, Donoghue considered in [**1**] the *corners*, which are the points of $C(A) \cap W(A)$ where $C(A)$ fails to have a unique tangent, and proved that they are eigenvalues of A. For normal operators, where we can use the spectral theorem, it is easy to prove that $\overline{W(A)}$ equals the convex hull of the spectrum.

Our proposal to generalize Donoghue's result is to consider points where $C(A)$ is *not round*, i.e., where the curvature is infinite. To be

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