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ARITHMETICAL CONSEQUENCES OF A SEXTUPLE PRODUCT IDENTITY

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ABSTRACT. The author derives a sixfold infinite-product identity in three complex variables. From this identity two formulas, one giving the number of representations of a given natural number by sums of four triangular numbers and the other giving the number of representations by sums of eight triangular numbers, are then deduced.

1. Introduction. The main result of this paper is the identity of the following theorem.

Theorem 1.1. *For each triple of complex numbers* a, b, x, *with* $a \neq 0$, $b \neq 0$ *and* $|x| < 1$,

$$\begin{aligned}(1.1) \quad \prod_1^\infty (1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1}) \\ (1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1}) \\ = \sum_{-\infty}^{\infty} x^{2m^2}a^{2m}\sum_{-\infty}^{\infty} x^{2n^2}b^{2n} \\ + x\sum_{-\infty}^{\infty} x^{2m(m+1)}a^{2m+1}\sum_{-\infty}^{\infty} x^{2n(n+1)}b^{2n+1}.\end{aligned}$$

Section 2 supplies a proof of this theorem. In Section 3 we apply the theorem to derive formulas for the numbers of representations of a given natural number by sums of four triangular numbers and by sums of eight triangular numbers. Our concluding remarks compare our formulas with two formulas of Jacobi for representations of numbers by sums of squares.

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