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OPERATORS ON C^*-ALGEBRAS INDUCED BY CONDITIONAL EXPECTATIONS

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ABSTRACT. This paper investigates the relationship between a unital C^*-algebra $\mathcal{A}$ and a C^*-subalgebra $\mathcal{B}$ which is the range of a conditional expectation operator on $\mathcal{A}$ by studying a certain algebra $\mathcal{D}$ of operators on $\mathcal{A}$. The investigation of $\mathcal{D}$ was suggested by previous work of A. Lambert and B. Weinstock in the case where the conditional expectation operators were the classical ones of probability theory.

The commutant of $\mathcal{D}$, the radical $\operatorname{Rad}\mathcal{D}$, the quotient $\mathcal{D}/\operatorname{Rad}\mathcal{D}$, the spectra of elements of $\mathcal{D}$ and the lattice of invariant subspaces for $\mathcal{D}$ are studied, as well as the questions of when $\mathcal{D}$ is closed in the norm and strong operator topologies.

Introduction. In [**5**] the relationship between a probability space (X, Σ, m) and a σ subalgebra Σ_1 of Σ is studied by using a certain algebra of bounded operators on $L^2(X, \Sigma, m)$. These operators are defined in terms of the classical conditional expectation $E(\,|\Sigma_1)$, and they have several natural analogues in which the classical conditional expectation is replaced by a conditional expectation operator defined on a C^*-algebra (see Section 1 below). The purpose of this paper is to study the relationship between a unital C^*-algebra $\mathcal{A}$ and a C^*-subalgebra $\mathcal{B}$ which is the range of a conditional expectation operator on $\mathcal{A}$ by investigating one such analogue which seems particularly natural from the perspective of the theory of associative operator algebras. In the noncommutative case our chosen analogue causes us to lose the relationship (which is present in [**5**]) to the notion of Banach-Lie algebra. That relationship is preserved, however, by other analogues of the operators in [**5**], which clearly deserve further study.

In Section 1 we define a nondegeneracy condition (called "type 0") for a C^*-subalgebra $\mathcal{B}$ which is modeled on an analogous property for σ subalgebras which plays a key role in [**5**]. We also introduce a modification of this condition ("restricted type 0") which is often more appropriate for the case that the subalgebra does not contain the

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