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DIOPHANTINE APPROXIMATION OF MATRICES

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ABSTRACT. Upper and lower bound results are given for the approximation of real matrices by quotients of integer matrices.

Introduction. In the theory of Diophantine approximations the approximated objects are usually numbers or functions. In this study we consider matrices as objects of which we investigate approximation properties. A typical question is the following. Let m and n be given positive integers. Let A be an $m \times m$ matrix with integer entries. Let $\sqrt[n]{A}$ be a matrix whose n-th power equals A. How well can $\sqrt[n]{A}$ be approximated by rational matrices if it is not rational itself? The question shows some essential differences with the classical cases. For example, it is not even obvious how to define the distance between two matrices. Furthermore, there may be infinitely many choices for $\sqrt[n]{A}$, since

$$\begin{pmatrix} 1 & \lambda \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for any choice of λ.

In Section 2 we introduce some notation. In Sections 3, 4 and 5, we shall deal with upper bounds for approximations. In the homogeneous case we prove an analogue of Dirichlet's theorem on simultaneous approximations. It will turn out that the choice of the distance function is essential for the quality of the bounds. Lattice basis reduction algorithms can be used for computing good approximations in polynomial time. In the inhomogeneous case we derive an analogue of Kronecker's theorem on simultaneous diophantine approximations. We further state an effective version of this result which follows from the work of Kannan and Lovász.

In Sections 6, 7 and 8 we study lower bounds for approximations. We show that a generalization of Roth's theorem holds for 2×2-matrices,

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